On the Convergence of the Unsymmetric Successive Overrelaxation (USSOR) Method

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ABSTRACT

Given a linear system with an H-matrix, a splitting of Varga's type is considered, and a convergence theorem for the unsymmetric successive overrelaxation (USSOR) method is obtained. Our results extend the known area of the convergence.

1. INTRODUCTION

We consider the system of linear equations

$$Ax = b \tag{1}$$

where $A \in C^{n,n}$ is a nonsingular complex matrix with nonzero diagonal elements, and $b \in C^n$ is a known vector.

One way to split the matrix A is to write

$$A = D(E - L - U), \tag{2}$$

where D is a diagonal matrix and L, U are strictly lower and strictly upper triangular matrices, respectively. Here, E denotes the $n \times n$ identity matrix.

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The unsymmetric successive overrelaxation method can be viewed as a two half iteration method. The first half iteration is the same as the SOR method, while the second half is the SOR method with different parameter and equations taken in reverse order, i.e.,

$$u^{n+1/2} = \mathscr{L}_\sigma u^n + (E - \sigma L)^{-1}\sigma b,$$

$$u^{n+1} = \mathscr{U}_\omega u^{n+1/2} + (E - \omega U)^{-1}\omega b, \tag{3}$$

where

$$\mathscr{L}_\sigma := (E - \sigma L)^{-1}[(1-\sigma)E + \sigma U],$$

$$U_\omega := (E - \omega U)^{-1}[(1-\omega)E + \omega L]. \tag{4}$$

From (3) and (4) we have

$$u^{n+1} = (E - \omega U)^{-1}[(1-\omega)E + \omega L](E - \sigma L)^{-1}$$

$$\times[(1-\sigma)E + \sigma U] + k,$$

$$k = \mathscr{U}_\omega(E - \sigma L)^{-1}\sigma b + (E - \omega U)^{-1}\omega b, \tag{5}$$

so the iteration matrix of the USSOR method can be written as

$$\mathscr{S}_{\sigma\omega} = \mathscr{L}_\sigma \mathscr{U}_\omega. \tag{6}$$

If $\sigma = \omega$ we get the symmetric successive overrelaxation method (SSOR) with iteration matrix $\mathscr{S}_{\sigma\sigma}$. The spectral radius of $\mathscr{S}_{\sigma\omega}$, $\rho(\mathscr{S}_{\sigma\omega})$, is obviously a function of σ, ω, and our aim in this paper is to derive intervals of convergence for the USSOR method, i.e., intervals where $\rho(\mathscr{S}_{\sigma\omega}) < 1$. The convergence domains which are obtained in this paper are not the best possible, like the results for the SSOR and USSOR methods in the papers of Neumaier and Varga [7] and Saridakis [10]. Determination of the exact convergence and divergence conditions should be the subject of further investigations. Some results on the SSOR and USSOR method are presented in Young [11], Alefeld and Varga [1], Krishna [5], and Martins and Krishna [6]. We consider problems where the matrix A is an H-matrix, and we extend some cited results for this class of matrices.

2. H-MATRICES

This class of matrices is very important in practice, because the discretization of many problems leads to a system of linear equations of the form (1), where the matrix A is an H-matrix.

In this paper, almost all notation used is the same as that adopted by Young [11]. Let us consider the matrix $A = [a_{ij}] \in C^{n,n}$ with complex elements. The space of all $n \times n$ matrices with complex elements and nonzero diagonal elements will be denoted by $C_{\pi}^{n,n}$.

For a matrix $A = [a_{ij}] \in C^{n,n}$, according to Varga [12], the comparison matrix $M(A) = [\alpha_{ij}] \in R^{n,n}$ is defined by

$$\alpha_{ii} := |a_{ii}|, \qquad 1 \leqslant i \leqslant n;$$

$$\alpha_{ij} := -|a_{ij}|, \qquad i \neq j, \quad 1 \leqslant i \leqslant n.$$

The space of equimodular matrices for the matrix A is the set

$$\Omega(A) = \left\{C = [c_{ij}] \in C^{n,n} : |c_{ij}| := |a_{ij}|, 1 \leqslant i,j \leqslant n\right\}.$$

Obviously, A and $M(A)$ belong to $\Omega(A)$.

For a matrix $A = [a_{ij}] \in C_{\pi}^{n,n}$, $n \geqslant 2$, with diagonal $D = \operatorname{diag}(a_{11}, \dots, a_{nn})$, its associated Jacobi matrix $J(A)$ is defined as $A = D[E - J(A)]$.

We are also going to need the following definitions and theorems from Varga [12]:

DEFINITION 1. A real $n \times n$ matrix $A = [a_{ij}]$ with $a_{ij} \leqslant 0$ for all $i \neq j$ is an M-matrix if A is nonsingular and $A^{-1} \geqslant 0$.

DEFINITION 2. A real $n \times n$ matrix $A = [a_{ij}]$ with $a_{ij} \leqslant 0$ for all $i \neq j$ is a Stieltjes matrix if A is symmetric and positive definite.

DEFINITION 3. For $n \times n$ real matrices A, M, and N, $A = M - N$ is a regular splitting of the matrix A if M is nonsingular and $M^{-1} \geqslant 0$, $N \geqslant 0$.

THEOREM 1. *If A is a Stieltjes matrix, then it is also an M-matrix.*

THEOREM 2. *If $A \geqslant 0$ is an $n \times n$ matrix, then the following are equivalent:*

(a) $\alpha > \rho(A)$;
(b) $\alpha E - A$ *is nonsingular and* $(\alpha E - A)^{-1} \geqslant 0$.

THEOREM 3. *If $A = M - N$ is a regular splitting of the matrix A and $A^{-1} \geqslant 0$, then*

$$\rho(M^{-1}N) < 1.$$

For an extension of the region of convergence for the USSOR method, we have also used the following theorem from Bohl [2].

THEOREM 4. *Let B be an M-matrix, $A = [a_{ij}] \geqslant B$, and $a_{ij} \leqslant 0$, $i \neq j$. Then A is also an M-matrix.*

DEFINITION 4. A matrix $A \in C^{n,n}$ is an H-matrix if $M(A)$ is an M-matrix.

Because of the great practical importance of H-matrices, they are divided into subclasses. Extension of the area of convergence for some subclasses will be the subject of further investigation.

3. CONVERGENCE OF THE USSOR METHOD

As we have mentioned above, the spectral radius of the matrix $\mathscr{S}_{\sigma\omega}$ is a function of the parameters σ, ω, and the following theorem gives the intervals of σ, ω where the USSOR method is convergent, i.e., its spectral radius $\rho(\mathscr{S}_{\sigma\omega})$ satisfies $\rho(\mathscr{S}_{\sigma\omega}) < 1$.

THEOREM 5. *Let $A = [a_{ij}] \in C_{\pi}^{n,n}$, $n \geqslant 2$. Then the following statements are equivalent:*

(a) *A is a nonsingular H-matrix.*
(b) *For all $C \in \Omega(A)$ and all $\sigma, \omega \in \mathscr{O}$ the USSOR method is convergent, where the set $\mathscr{O}$ is defined as*

$$\sigma \in \left(-\frac{1-\rho}{2\rho}, \frac{\rho+1}{2\rho}\right), \qquad \textit{with} \quad \rho = \rho(|L| + |U|),$$

$$\omega \in \left(\max\left\{\frac{|1-\sigma| + |\sigma|\rho - 1}{|1-\sigma| + \rho(|\sigma|+1)}, \frac{|1-\sigma| + |\sigma|\rho - 1}{|1-\sigma|(1-\rho)}\right\},\right.$$

$$\left.\min\left\{\frac{1 + |1-\sigma| + \rho|\sigma|}{\rho(1+|\sigma|) + |1-\sigma|}, \frac{1+|1-\sigma| - \rho|\sigma|}{|1-\sigma|(1+\rho)}\right\}\right).$$

Proof. (a) → (b): If A is an H-matrix, then $M(A)$ is an M-matrix, so that

$$\rho(J(M(A))) = \rho(|J(C)|) < 1$$

for all $C \in \Omega(A)$. The matrix C can be split as

$$C = D(E - L - U),$$

where D is a diagonal matrix, and L, U are strictly lower and strictly upper triangular matrices. So the matrix $\mathscr{S}_{\sigma\omega}$ is given, from [6], as

$$\mathscr{S}_{\sigma\omega} = (E - \omega U)^{-1}[(1 - \omega)E + \omega L](E - \sigma L)^{-1}[(1 - \sigma)E + \sigma U].$$

As

$$[(1 - \omega)E + \omega L](E - \sigma L)^{-1} = (E - \sigma L)^{-1}[(1 - \omega)E + \omega L],$$

we can write

$$\mathscr{S}_{\sigma\omega} = (E - \omega U)^{-1}(E - \sigma L)^{-1}[(1 - \omega)E + \omega L][(1 - \sigma)E + \sigma U].$$

With the obvious inequalities:

$$|(E - \omega U)^{-1}| \leqslant (E - |\omega|\,|U|)^{-1},$$

$$|(E - \sigma L)^{-1}| \leqslant (E - |\sigma|\,|L|)^{-1},$$

we get

$$|\mathscr{S}_{\sigma\omega}| \leqslant \mathscr{S},$$

where

$$\mathscr{S} := (E - |\omega|\,|U|)^{-1}(E - |\sigma|\,|L|)^{-1}(|1 - \omega|E + |\omega|\,|L|)$$
$$\times(|1 - \sigma|E + |\sigma|\,|U|).$$

Now, consider the matrix

$$B = \frac{1 - |1 - \omega|\,|1 - \sigma|}{|\omega| + |\sigma|\,|1 - \omega|} E - \frac{|\sigma| + |\omega|\,|1 - \sigma|}{|\omega| + |\sigma|\,|1 - \omega|} |L| - |U|$$

and its splitting

$$B = M - N,$$

where

$$M := \frac{1}{|\omega| + |\sigma|\,|1 - \omega|} (E - |\sigma|\,|L|)(E - |\omega|\,|U|)$$

and

$$N := \frac{1}{|\omega| + |\sigma|\,|1 - \omega|} (|1 - \omega| E + |\omega|\,|L|)(|1 - \sigma| E + |\sigma|\,|U|).$$

It is clear that

$$\mathscr{S} = M^{-1} N,$$

and as $M^{-1} \geqslant 0$, $N \geqslant 0$, this splitting of the matrix B is regular.

Let us now consider two cases:

$$1 - |1 - \omega|\,|1 - \sigma| > 0,$$

$$\frac{|\sigma| + |\omega|\,|1 - \sigma|}{|\omega| + |\sigma|\,|1 - \omega|} \leqslant 1,$$

$$\frac{|\omega| + |\sigma|\,|1 - \omega|}{1 - |1 - \omega|\,|1 - \sigma|} \rho < 1, \tag{i}$$

and

$$1 - |1 - \omega|\,|1 - \sigma| > 0$$

$$\frac{|\sigma| + |\omega|\,|1 - \sigma|}{|\omega| + |\sigma|\,|1 - \omega|} \geqslant 1,$$

$$\frac{|\sigma| + |\omega|\,|1 - \sigma|}{1 - |1 - \omega|\,|1 - \sigma|} \rho < 1. \tag{ii}$$

These two groups of conditions define the set $\mathscr{O}$.

If we choose σ, ω so that case (i) is satisfied, then

$$B \geqslant \frac{1 - |1 - \omega|\,|1 - \sigma|}{|\omega| + |\sigma|\,|1 - \omega|}E - (|L| + |U|) = W,$$

and $W^{-1} \geqslant 0$. Thus according to Theorem 4, we can conclude that $B^{-1} \geqslant 0$. In case (ii) we have

$$B \geqslant \frac{1 - |1 - \omega|\,|1 - \sigma|}{|\omega| + |\sigma|\,|1 - \omega|}E - \frac{|\sigma| + |\omega|\,|1 - \sigma|}{|\omega| + |\sigma|\,|1 - \omega|}(|L| + |U|) = W,$$

and $W^{-1} \geqslant 0$, so that by Theorem 4, we also have $B^{-1} \geqslant 0$.

Now we have proved that for all $(\sigma, \omega) \in \mathscr{O}$ the matrix B is an M-matrix. Thus we can apply Theorem 3, which gives us

$$\rho(\mathscr{S}) < 1.$$

By Theorem 2.8, p. 89 of [12], we can conclude that

$$\rho(\mathscr{S}_{\sigma\omega}) \leqslant \rho(|\mathscr{S}_{\sigma\omega}|) \leqslant \rho(\mathscr{S}) < 1,$$

and so the USSOR method is convergent for all $(\sigma, \omega) \in \mathscr{O}$.

(b) → (a): The Proof is analogous to the proof of the corresponding theorem from [6]. ∎

For an H-matrix A with $\rho(J(\mathscr{M}(A))) = 0.4$ the set $\mathscr{O}$ is presented in Figure 1.

The set $\mathscr{O}$ is symmetric about the line $\omega = \sigma$, and in the case of the SSOR method, the result of Alefeld and Varga [1] is obtained. Unfortunately, this is not the exact area of the convergence for the USSOR method applied to H-matrices. Simple examples can be given which show that the USSOR method is convergent for parameters (σ, ω) which do not belong to the set $\mathscr{O}$. Determining the exact convergence conditions, analogous to those obtained for the SSOR method in the paper of Neumann and Varga [7] and for the USSOR method applied to p-cyclic matrices by Saridakis [10], should be the subject of further investigation.

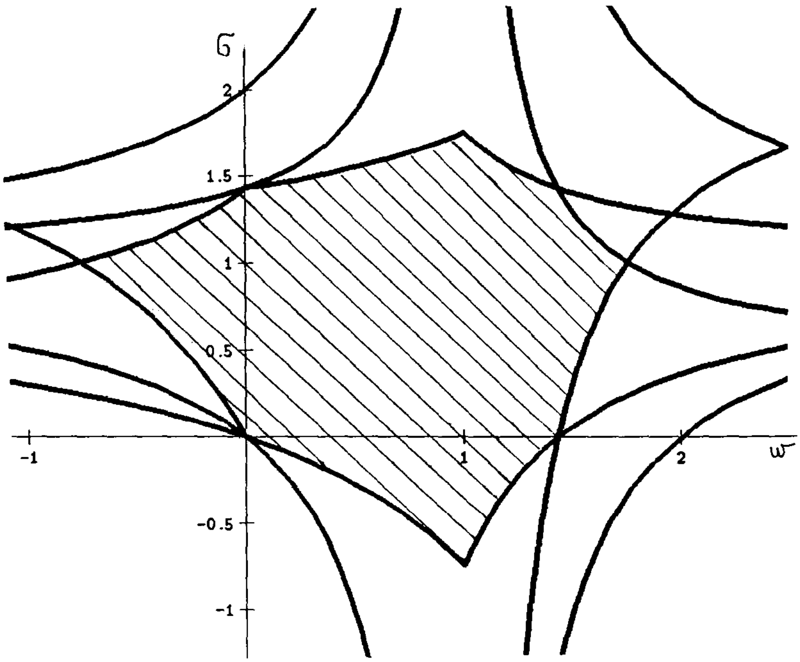

FIG. 1.

COROLLARY 1. *Let A be a strictly diagonally dominant matrix or an irreducible diagonally dominant matrix. Then the USSOR method is convergent for all $(\sigma, \omega) \in \mathscr{P}$.*

For a matrix $A \in C^{n,n}$ we introduce the following notation:

$$N = \{1, 2, \ldots, n\}, \qquad N(i) = N \setminus \{i\},$$

$$P_i(A) = \sum_{j \in N(i)} |a_{ij}|, \qquad Q_i(A) = \sum_{j \in N(i)} |a_{ji}|,$$

$$P_{i,\alpha}(A) = \alpha P_i(A) + (1 - \alpha) Q_i(A),$$

$$Q_i^*(A) = \max_{j \in N(i)} |a_{ij}|, \qquad Q_i^r(A) = \max_{t_r \in \Theta_r} \sum_{j \in t_r} |a_{ji}|,$$

where $i, r \in N$, $\alpha \in [0, 1]$, and Θ_r is a set of all choices $t_r = \{i_1, i_2, \dots, i_n\}$ of different indices from N.

Using a characterization of H-matrices given in the paper by Cvetković and Herceg [3], we have the following

COROLLARY 2. *Let A be a matrix from $C^{n,n}$ which satisfies at least one of the following conditions for some $\alpha \in [0, 1]$:*

1. $|a_{ii}| > P_{i,\alpha}(A)$, $i \in N$.
2. $|a_{ii}| > P_i^{\alpha} Q_i^{1-\alpha}(A)$, $i \in N$.
3. $|a_{ii}|\,|a_{jj}| > P_i(A) P_j(A)$, $i \in N$, $j \in N(i)$.
4. $|a_{ii}|\,|a_{jj}| > P_i^{\alpha}(A) Q_i^{1-\alpha} P_j^{\alpha}(A) Q_j^{1-\alpha}(A)$, $i \in N$, $j \in N(i)$.
5. $|a_{ii}| > P_{i,\alpha}(A)$ *or* $|a_{ii}| + \Sigma_{j \in J} |a_{jj}| > Q_i(A) + \Sigma_{j \in J} Q_j(A)$, $J := \{i \in N : |a_{ii}| \leqslant Q_i(A)$.
6. $|a_{ii} > \min(P_i(A), Q_i^*(A))$, $i \in N$, *and* $|a_{ii}|\,|a_{jj}| > P_i(A) + P_j(A)$, $i \in N$, $j \in N(i)$.
7. $|a_{ii}| > Q_i^p(S + T)$, $i \in N$, *and* $\Sigma_{j \in t_p} |a_{ii}| > \Sigma_{j \in t_p} P_i(A)$, $t_p \in \theta_p$, *for some* $p \in N$.
8. *There exists an* $i \in N$ *such that* $|a_{ii}|[|a_{jj}| - P_j(A) + |a_{ji}|] > P_i(A)|a_{ji}|$, $j \in N(i)$.

Then the USSOR method is convergent for all $(\sigma, \omega) \in \mathscr{O}$.

If a matrix A is Hermitian and positive definite, we have the well-known result (see [11]), that the USSOR method is convergent for all $\sigma \in (0, 2)$, $\omega \in (0, 2)$. Using this and results of Theorem 5, we have

COROLLARY 3. *Let A be a Stieltjes matrix. Then the USSOR method is convergent for* $\sigma \in (0, 2)$, $\omega \in (0, 2)$, *and* $(\sigma, \omega) \in \mathscr{O}_1 \cup \mathscr{O}_2$, *where*

$$\mathscr{O}_1 : \begin{cases} 0 < \sigma < \dfrac{2}{1 + \rho}, \\[2ex] 0 \geqslant \omega \geqslant \dfrac{|1 - \sigma| + \sigma\rho - 1}{|1 - \sigma| + \rho(1 + \sigma)}, \end{cases}$$

$$\mathscr{O}_2 : \begin{cases} 0 \geqslant \sigma > -\dfrac{1 - \rho}{2\rho}, \\[2ex] \dfrac{(1 - \sigma) - \rho\sigma - 1}{(1 - \sigma)(1 - \rho)} < \omega < \dfrac{1 + (1 - \sigma) + \rho\sigma}{(\rho + 1)(1 - \sigma)}. \end{cases}$$

4. NUMERICAL RESULTS

We have solved the system arising from the discretization of the Dirichlet problem on a uniform square grid. This problem is known as a model problem, and it has been used for testing the efficiency of iteration methods by many authors; see [12] for detailed discussion of this problem. For a given positive integer n, we have $(n-1)^2$ unknowns x_{ij} and $(n-1)^2$ linear equations:

$$4x_{ij} - x_{i-1,j} - x_{i,j+1} - x_{i+1,j} - x_{i,j-1} = 0,$$

$i = 1, 2, \dots, n-1$, $j = 1, 2, \dots, n-1$, with the boundary condition

$$0 = x_{kn} = x_{nk} = x_{0k} = x_{k0}, \qquad k = 1, 2, \dots, n.$$

If we take the natural ordering of mesh points x_{ij}, the matrix which arises from this discretization is a Stieltjes matrix. The starting values are taken as in Young [11], i.e.,

$$x_{ij} = 1, \qquad i = 1, \dots, n-1, \quad j = 1, \dots, n-1,$$

and the iterative process was terminated when the condition

$$\max_{i,j} |x_{ij}| \leqslant 10^{-6}$$

was satisfied. Let k be the number of iterations required to satisfy the termination criterion. Table 1 shows the results of our numerical experiments. We have tested the SOR, SSOR, and USSOR methods for $n = 20$. In this table total iterations were counted. Although one complete iteration of

TABLE 1

USSOR		SSOR		SOR	
(σ, ω)	k	σ	k	σ	k
(0.500, 1.779)	55	0.500	466	0.500	927
(0.813, 1.837)	57	0.813	228	0.813	449
(0.938, 1.712)	42	0.938	178	0.938	347
(1.000, 1.712)	40	1.000	158	1.000	305
(1.188, 1.739)	38	1.188	109	1.188	206
(1.312, 1.739)	37	1.312	86	1.312	156
(1.439, 1.987)	51	1.439	66	1.439	112
(1.635, 1.813)	40	1.635	46	1.632	53
(1,697, 1.250)	39	1.697	44	1.697	61
(1.822, 1.500)	39	1.822	55	1.822	45

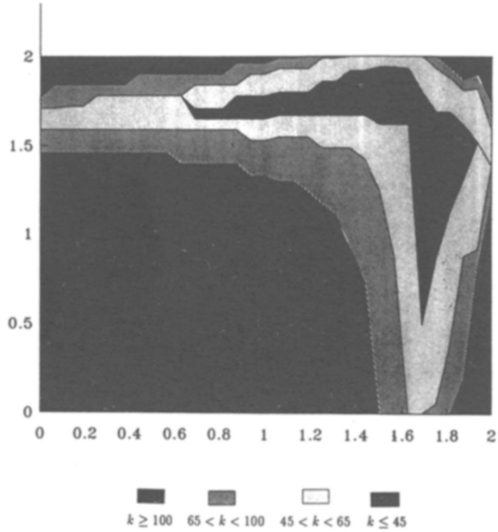

FIG. 2.

the USSOR method requires approximately twice as much work as one iteration of the SOR method, the results show that it can be faster in some cases, i.e., the number of USSOR iterations can be less than one-half the number of SOR iterations.

It is obvious that ω and σ have a great influence on the convergence speed of the USSOR method. Figure 2 shows the areas of different speed for the model problem and $n = 20$.

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